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A modified Green's function for the internal gravitational wave equation in a layer of a stratified medium with a constant shear flow☆

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ABSTRACT

The construction of a modified Green's function for the internal gravitational wave (IGW) equation in a layer of a stratified medium when there are constant mean shear flows is considered and the basic properties of the corresponding eigenvalue problems and the modified eigenfunctions and eigenvalues are investigated. It is shown that each mode of the modified Green's function consists of a sum of three terms describing (1) the IGWs that propagate from the source, (2) the effects of a time varying source, localized in a certain neighbourhood of it, and (3) the effects of the displacement of the fluid (an internal discontinuity) caused by the source. The resulting expressions are analysed out for a constant and oscillating source of the generation of IGWs in which each of the terms of Green's function is represented in the form of simple quadratures.

1. Formulation of the problem

The majority of problems concerned with the mathematical modelling of the generation of internal gravitational waves (IGWs) by different non-localized perturbations when there are mean shear flows are solved in a linear formulation, that is, under the assumption that the amplitude of the wave motions is small compared with the wavelength.[1–6] As a rule, the problem of constructing Green's function for the IGW equation is considered in this formulation, which makes it possible to describe the wave fields excited in the case of the motion of a point source of perturbations in a stratified medium with an arbitrary density distribution throughout its depth. Even within the limits of linear models, the resulting equations in the form of multiple quadratures are quite unusual. In the general case of wave generation by arbitrary non-local sources of perturbations, the solution for all the components of the wave fields is expressed in terms of Green's function for the IGW equation and its asymptotic representations.[1–6]

Green's function Γ for the IGW equation, when there are mean shear flows in a layer $-\boldsymbol{H}<z<0$ of a stratified medium,[1,2] is considered next, namely,

$$\frac{D^2}{Dt^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Gamma-\frac{D}{Dt}\left(\frac{\partial^2 V_1}{\partial z^2}\frac{\partial}{\partial x}+\frac{\partial^2 V_2}{\partial z^2}\frac{\partial}{\partial y}\right)\Gamma+N^2(z)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Gamma=$$
$$=\delta(t)\delta(x)\delta(y)\delta(z-z');\qquad \frac{D}{Dt}=\frac{\partial}{\partial t}+V_1\frac{\partial}{\partial x}+V_2\frac{\partial}{\partial y} \tag{1.1}$$

where $N(z)$ is the Väisälä–Brunt frequency, V_1 and V_2 are the components of the flow velocity $\mathbf{V}=\{V_1, V_2, 0\}$ for a certain level z and z' is the immersion depth of the point source.

The boundary and initial conditions are taken in the form

$$z=0,-H:\quad \Gamma=0;\quad t<0:\quad \Gamma\equiv 0 \tag{1.2}$$

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Using a Fourier transformation with respect to the variables t, x and y

$$\Gamma = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \exp(i(\lambda x + \mu y - \omega t))G(\omega, \lambda, \mu, z, z')d\omega$$

we obtain the boundary-value problem

$$LG = -\delta(z-z'); \qquad z = 0, -H: \quad G = 0$$

$$L = (\omega - f)^2\frac{\partial^2}{\partial z^2} + k^2[N^2 - (\omega - f)^2] + \frac{\partial^2 f}{\partial z^2}(\omega - f), \quad f \equiv \lambda V_1(z) + \mu V_2(z), \quad k^2 = \lambda^2 + \mu^2$$

for the Fourier transform of Green's function G.

The expression for Green's function, that takes account of the initial condition (1.2), has the form[1,2]

$$\Gamma = G_m(t, x, y, z, z') + \sum_{n=1}^{\infty}\frac{1}{2\pi^2}\text{Im}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} \exp(i(\lambda x + \mu y - \omega_n(\lambda, \mu)t))\frac{\varphi_n(z)\varphi_n(z')}{d_n(\omega_n - f(z'))^2}d\mu$$

$$d_n = \left.\frac{\partial\varphi_n}{\partial\omega}\frac{\partial\varphi_n}{\partial z}\right|_{z=-H,\, \omega=\omega_n}$$

where $\varphi_n(z)$ are eigenfunctions, $\omega_n(\lambda, \mu)$ are the characteristic frequencies of the operator L, and the function G_m is the contribution of the continuous spectrum of the operator L. For large values of $|x|$, $|y|$ and $|t|$ the wave zone is bounded by two curves, the leading and rear fronts, and, when $\mathbf{V}$ = const, the contribution from the continuous spectrum and the rear front vanish.[1,2]

2. The modified vertical eigenvalue problem

We next consider the representation of Green's function, when $\mathbf{V}$ = const, in a modified form which allows the spatial structure of the excited wave fields to be more completely revealed both at considerable distances as well as in the immediate vicinity of the sources of IGWs generation. The conventional method of solving of the boundary value problem is to expand the function G in a complete set of linearly independent functions which are the solutions of the corresponding homogeneous problem.[1] As such a set, we will take the solution of the following boundary value problem

$$\frac{\partial^2\psi_n(z,\chi)}{\partial z^2} + \chi\left[\frac{N^2(z)}{\nu_n(\chi)} - 1\right]\psi_n(z,\chi) = 0; \quad z = 0, -H: \quad \psi_n = 0 \tag{2.1}$$

For positive values of χ, the solution of problem (2.1) describes vertical modes of oscillations of the particles in the stratified medium when there are no flows. For negative values of χ, the solution of problem (2.1) describes the oscillations of the fluid in domains of internal discontinuities. In order to distinguish problem (2.1) from the problem conventionally used in which $\chi > 0$ always,[1–3,6] we shall call it the modified problem. The behaviour of the eigenfunctions $\psi_n(z, \chi)$ when z is varied and, correspondingly, the distribution of the zeroes of these functions along the z axis is determined by the sign of the factor in front of the function $\psi_n(z, \chi)$ in problem (2.1). When $\chi \geq 0$, $\nu_n \geq 0$ or $\chi \leq 0$, $\nu_n \leq 0$, the zeroes of the function ψ_n are concentrated in the neighbourhood of those z where the value of $N(z)$ is a maximum. When $\chi < 0$, $\nu_n > 0$, the zeroes of the function ψ_n are concentrated in the domain with the minimum value of $N(z)$. There are obviously no oscillatory solutions when $\chi > 0$, $\nu_n < 0$. The monotonicity of the dispersion curves $\nu_n(\chi)$ for any χ also follows from the formulation of problem (2.1). Taking account of the correspondence between the number of zeroes of a function ψ_n and its number n, from the solution of problem (2.1) it is possible to determine the five singular points of the dispersion curves $\nu_n(\chi)$:

$$1)\ \nu_n(\chi) \to \max_z N^2(z),\ \chi \to +\infty; \quad 2)\ \nu_n(0) = 0; \quad 3)\ \nu_n(\chi) \to -\infty,\ \chi \to -\left(\frac{\pi n}{H}\right)^2 + 0;$$

$$4)\ \nu_n(\chi) \to +\infty,\ \chi \to -\left(\frac{\pi n}{H}\right)^2 - 0; \quad 5)\ \nu_n(\chi) \to \min_z N^2(z),\ \chi \to -\infty.$$

3. The modified Green's function

Next, by representing the solution of problem (1.1), (1.2) in the form of an expansion in the set of eigenfunctions

$$G = \sum_{n=1}^{\infty} p_n\psi_n(z, k^2)$$

and finding p_n by convolution of the function G with $\psi_n(z, k^2)$, we obtain an expression for the modified Green's function in the form

$$\Gamma = \sum_{n=1}^{\infty}\Gamma_n, \quad \Gamma = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}\exp(i(\lambda x+\mu|y|-\omega t))\frac{\nu_n(k^2)b_n(k^2)}{k^2[(\omega+i\varepsilon-f)^2-\nu_n(k^2)]}d\mu$$

$$b_n(k^2) = \frac{\psi_n(z,k^2)\psi_n^*(z,k^2)}{B_n(k^2)}, \quad B_n(k^2) = \int_{-H}^{0}N^2(z)\left|\psi_n(z,k^2)\right|^2dz \tag{3.1}$$

Assuming that the x axis is always directed against the flow velocity $\left(f = -\lambda V,\ V = \sqrt{V_1^2+V_2^2}\right)$, we obtain the equation for the poles of the integrand $\nu_n = (\omega + i\varepsilon - f)^2$ or

$$k^2\tilde{N}_n^2(k^2) = (\omega+i\varepsilon+\lambda V)^2\left[k^2+\left(\frac{\pi n}{\tilde{H}_n(k^2)}\right)^2\right]$$

$$\tilde{N}_n(k^2) = \left[\frac{B_n(k^2)}{A_n(k^2)}\right]^{1/2}, \quad \tilde{H}_n(k^2) = \left[A_n(k^2)\left(\int_{-H}^{0}\left|\frac{1}{\pi n}\frac{\partial\psi_n(z,k)^2}{\partial z}\right|dz\right)^{-1}\right]^{1/2},$$

$$A_n(k^2) = \int_{-H}^{0}\left|\psi_n(z,k^2)\right|^2dz$$

which enables us to evaluate one of the integrals in expression (3.1).

When $t > 0$, on integrating with respect to ω in expression (3.1) for Γ_n, we have

$$\Gamma_n = -\frac{1}{2\pi}\int_{-\infty}^{\infty}J_0(kR)\sin(t\sqrt{\nu_n(k^2)})b_n(k^2)\frac{\sqrt{\nu_n(k^2)}}{k^2}dk, \quad R = [(x+Vt)^2+y^2]^{1/2} \tag{3.2}$$

where J_0 is a zero-order Bessel function.

When $t < 0$, the integration contour is closed in the upper half-plane, which leads to the equality $\Gamma_n = 0$ in accordance with the initial condition (1.2). In a system of coordinates moving together with the flow ($x \to x - Vt$), expression (3.2) possesses circular symmetry and is identical to the result obtained earlier[1] for Green's function.

In practical applications, such as, for example, in the problem of the generation of IGWs by means of pulsating sources, it is often convenient to use the spectral density $\tilde{\Gamma}_n$ of Green's function:

$$\Gamma_n = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Gamma}_n d\omega$$

In this case, it is necessary to integrate with respect to λ or with respect to μ in expression (3.1). Integration with respect to λ was used earlier[1] in relation to the problem of generating IGWs by means of a source of constant intensity ($\omega = 0$), where Green's function was represented in the form of a series in eigenfunctions of the usual boundary value problem. In this case, the modified problem (2.1) is identical to the conventional problem when $\chi > 0$. As a result, only the real poles $\lambda = \pm\lambda_n(\mu^2)$ are taken into account. This does not change the asymptotic representations of Green's function for large t, $|x|$, $|y|$ that is solely determined by the real poles.

Use of the modified problem determines that, apart from the pairs of real poles, there are also pairs of pure imaginary poles $\lambda_n(\mu^2)$. In fact, for poles which are unshifted by the addition of $i\varepsilon$, we have the relation

$$\lambda_n^2(\mu^2) = \frac{1}{2}\left[\frac{\tilde{N}_n(k^2)}{V^2}-\mu^2-\left(\frac{\pi n}{\tilde{H}_n(k^2)}\right)^2\right] \pm \sqrt{\left[\frac{\tilde{N}_n(k^2)}{V^2}-\mu^2-\left(\frac{\pi n}{\tilde{H}_n(k^2)}\right)^2\right]^2+4\mu^2\frac{\tilde{N}_n(k^2)}{V^2}}$$

$$k^2 = \mu^2+\lambda_n^2(\mu^2) \tag{3.3}$$

Since, by definition, the quantities $\tilde{N}_n(k^2)$ and $\tilde{H}_n(k^2)$ are real, the realness of $\lambda_n^2(\mu^2)$ follows from equality (3.3). If the Väisälä–Brunt frequency depends only slightly z, the quantities $\tilde{N}_n(k^2)$ and $\tilde{H}_n(k^2)$ are independent of k^2 in the first approximation and Eq. (3.3) directly determines the poles $\lambda_n = \lambda_n(\mu^2)$. The displacement of the real poles into the complex plane due to the addition of $i\varepsilon$ is determined from the expansion

$$\lambda_n = \lambda_n\big|_{\varepsilon=0}+i\varepsilon\frac{\partial\lambda_n}{\partial\varepsilon}\bigg|_{\varepsilon=0}+\ldots; \quad \frac{\partial\lambda_n}{\partial\varepsilon} = V(\nu_n'(k^2)-V^2)^{-1}$$

When $\omega = 0$, the spectral density of Green's function can be represented in the form

$$\tilde{\Gamma}_n = \frac{1}{(2\pi)^2}\int_{0}^{\infty}d\mu\int_{-\infty}^{\infty}\exp(i(\lambda x+\mu|y|))\frac{\nu_n(k^2)b_n(k^2)}{k^2[(\lambda V+i\varepsilon)^2-\nu_n(k^2)]}d\lambda + \text{c.c.}$$

Hence and below, c. c. is the corresponding complex-conjugate term.

Next, assuming that the functions $\nu_n(k^2)$ and $b_n(k^2)$ are analytically extended into the domain of complex λ, we evaluate the integral with respect to this variable and obtain

$$\tilde{\Gamma}_n = -\frac{1}{2\pi}\int_0^\infty \frac{\sin(\lambda_n(\mu^2)|x|)}{\lambda_n(\mu^2)} I_n d\mu + \frac{1}{2\pi}\int_0^\infty \frac{\exp(-|\lambda_n(\mu^2)x|)}{2|\lambda_n(\mu^2)|} I_n d\mu + \text{c.c.}$$

$$I_n = \exp(i\mu|y|)\frac{\nu_n(k^2)b_n(k^2)}{k^2[V^2-\nu_n'(k^2)]}, \quad k^2 = \mu^2+\lambda_n^2(\mu^2), \quad \nu_n(k^2) = V^2\lambda_n^2(\mu^2) \tag{3.4}$$

The first term in sum (3.4) corresponds to positive values of $\lambda_n^2(\mu^2)$ and the second to negative values, whence it follows that the integral with positive $\lambda_n^2(\mu^2)$ determines the wave component of the field generated by the source.

Taking account of the fact that the quantities $\nu_n(k^2)$ and $\nu_n'(k^2)$ are bounded when $\lambda_n^2(\mu^2) \geq 0 (k^2 = \mu^2 + \lambda_n^2(\mu^2) \geq 0)$:

$$\nu_n(k^2) < \max_z N^2(z), \quad \nu_n'(k^2) \leq \nu_n'(0) = \frac{\tilde{N}^2(0)\tilde{H}_n^2(0)}{\pi^2 n^2} \equiv M_n^2, \quad \lim_{\mu\to\infty}\nu_n'(k^2) = 0$$

we obtain that a wave field always exists in the domain $x < 0$. In the domain $x > 0$, only a finite number of IGW modes exist, the numbers of which n do not exceed the quantity M_n.

Henceforth, for simplicity, we shall always assume that the condition $M_n < 1$ is satisfied. The asymptotics of the wave component of $\tilde{\Gamma}_n(\omega = 0)$ for large values of $|x|$, $|y|$ correspond to the results obtained earlier.[1] Because of the presence of the factor 731e, the integral with negative values of $\lambda_n^2(\mu^2)$ in expression (3.4) describes a rapidly decaying component Γ_n as $\exp(-|\lambda_n(\mu^2)x|)$ increases. If the wave field vanishes when there is no stratification ($N(z)=0$) since the quantity $\mu_n^2(\mu^2)$ is only non-negative at the point $\mu = 0$, the rapidly decaying term in integrals (3.4) is non-zero and thereby describes the effects of the displacement of the stratified medium by the source. We also note that, unlike the wave term, the integrand of the second term in expression (3.4) has singularities in certain domains of integration. In the more general case, when $\omega \neq 0$, these singularities are only removed when the internal wave fields from a source, which is extended along the y axis, are calculated: the well-known problem of the infinity of the energies of IGWs which are emitted by a point source.[1,2]

The functions $\tilde{\Gamma}_n$ for any x and y, which are more acceptable for analysis and calculation of the expression, can be obtained by calculating the quadratures in expression (3.4) with respect to the variable μ.

We now represent expression (3.4) in the form

$$\tilde{\Gamma}_n = \frac{1}{(2\pi)^2}\int_0^\infty d\lambda \int_{-\infty}^\infty \exp(i(\lambda x + \mu|y| - \omega(t + xV^{-1})))\frac{\nu_n(k^2)b_n(k^2)}{k^2[(\lambda V + i\varepsilon)^2 - \nu_n(k^2)]}d\mu + \text{c.c.}$$

$$k^2 = (\lambda - \omega V^{-1})^2 + \mu^2 \tag{3.5}$$

Then, the poles $\mu_n(\lambda, \omega)$ of the integrand in equality (3.5) satisfy the equation $\nu_n(k^2) = (\lambda V + i\varepsilon)^2$, or

$$\mu_n^2(\lambda, \omega) = -(\lambda - \omega V^{-1})^2 + \left(\frac{\pi n}{\tilde{H}_n(k^2)}\right)^2\left(\frac{\tilde{N}_n(k^2)}{\tilde{N}_n(k^2) - (\lambda V + i\varepsilon)^2} - 1\right)$$

$$k^2 = (\lambda - \omega V^{-1})^2 + \mu_n^2(\lambda, \omega)$$

The realness of the quadratures of the unshifted ($\varepsilon = 0$) poles $\mu_n^2(\lambda, \omega)$ for any λ and ω follows from this. Since the function $\nu_2 = \nu_n(k^2)$ is unbounded (this is one of the main differences between the modified problem and the conventional problem), this equation is always solvable for $\mu_n^2(\lambda, \omega)$.

We introduce the following notation

$$m = \min_z N(z), \quad M = \max_z N(z)$$

Depending on to the behaviour of the function ν_n, the following situations arise:

when $\lambda V < m$, there is a single positive value of the function ν and, consequently, a single pair of poles $\mu_n(\lambda, \omega)$.
when $m < \lambda V < m$, a single positive and, possibly, a single negative value of the function ν_n exists and, correspondingly, there are two (pure real and pure imaginary) pairs of poles $\mu_n(\lambda, \omega)$ (it is possible that both pairs are pure imaginary).
when $m < \lambda V$, a single negative value of the function ν_n exists, that is, a single pair of pure imaginary poles $\mu_n(\lambda, \omega)$.

Assuming that the integrand in expression (3.5) is analytically extended into the domain of complex values, we evaluate the integral with respect to μ. Taking account of the displacement of the poles from the real axis and closing the contour of integration in the upper half-plane, we have

$$\tilde{\Gamma}_n = (\tilde{\Gamma}_n^+ + \tilde{\Gamma}_n^0 + \tilde{\Gamma}_n^-)\exp(-i\omega(t + xV^{-1})) \tag{3.6}$$

$$\tilde{\Gamma}_n^{\pm} = \frac{1}{4\pi}\int_0^M D_n^{\pm} d\lambda + \text{c.c.}, \quad \mu_n^2 = \begin{cases} (\mu_n^+)^2 \geq 0 \\ (\mu_n^-)^2 \leq 0 \end{cases}$$

$$\tilde{\Gamma}_n^0 = \frac{1}{4\pi}\int_m^\infty D_n^- d\lambda + \text{c.c.}, \quad \mu_n^2 = (\mu_n^0)^2 \leq 0 \tag{3.7}$$

$$D_n^{\pm} = \exp(i\lambda x \pm |\mu_n(\lambda,\omega)||y|)\psi_n(z,k^2)\psi_n^*(z,k^2)(|\mu_n(\lambda,\omega)|(\lambda^2V^2 - \tilde{N}^2{}_n(k^2))A_n(k^2))^{-1} \tag{3.8}$$

The right-hand side of expression (3.8) contains singularities at points λ, where $|\mu_n(\lambda, \omega)| = 0$ and $\lambda^2V^2 = \tilde{N}_n(k^2)$. If the source is a point source, the quadratures (3.6) and (3.7) describe the field of the IGWs generated by it, and, at the same time, the singularities in the integrands remain. Real physical sources of disturbances in natural stratified media (an ocean or the atmosphere) always have a spatial extension, and corresponding regularization factors, which remove singularities of the above-mentioned types,[1,3] therefore appear in the integrands when calculating IGWs from non-local sources of disturbances.

4. The phase structure of the modified Green's function

The term $\tilde{\Gamma}_n^+$ determines the wave component of the modified Green's function, the asymptotics of which are determined for large values of $|x|$, $|y|$ by the points of stationary phase. The lines of constant phase, describing the wave fronts, are given in this case by the equation

$$xV^{-1}\sqrt{\nu_n(k^2)} + yV^{-1}\sqrt{k^2V^2 - (\omega - \sqrt{\nu_n(k^2)})^2} - (t + xV^{-1})\omega = \text{const}$$

Direct calculation gives a wave pattern, close to the standard Kelvin wave wedge, for the lines of constant phase. The wave pattern is symmetrical about the x axis and its boundary can be defined as

$$\max|y|x^{-1} = \max_k\{d_n(k)(V^2 + e_n(k)C_n^g(k))^{-1}\}$$

$$d_n(k) = C_n^g(k)\sqrt{V^2 - e_n^2(k)}, \quad e_n(k) = \omega k^{-1} - C_n^f(k)$$

where $C_n^g(k) = \partial\sqrt{\nu_n(k^2)}/\partial k$ and $C_n^f(k) = \sqrt{\nu_n(k^2)}/k$ are the group and phase velocities of a plane IGW with wave number k. In a system of coordinates associated with the source, the velocity of the wave zone S in a tangential direction is given by the expression (θ is the apex angle of the wave wedge)

$$S(\omega) = \max_k\left\{d_n(k)((C_n^g(k))^2 + V^2 + 2e_n(k)C_n^g(k))^{-1/2}\right\}$$

When $\omega = 0$, the rate of displacement of the wave front is equal to

$$S(0) = C_n^g(0)\sqrt{1 - (C_n^f(0))^2V^{-2}}$$

(we take account of the equality $C_n^g(0) = C_n^f(0)$). Hence, the rate of displacement of the wave front does not exceed the maximum group velocity of an individual IGW mode.[1]

If $\omega \neq 0$, the rate $S(\omega)$ will obviously be equal for $\omega = +|\omega_0|$ and $\omega = -|\omega_0|$: a situation which is encountered when constructing of the field of the IGWs from oscillating sources. In this case, we have two wave zones, superimposed on one another, the boundaries of which are not the same, and, here, the amplitudes of the IGWs in each zone are modulated by the frequency of the oscillation.

The term $\tilde{\Gamma}_n^0$ rapidly decays as $|y|$ increases and, at the same time, the eigenfunctions $\psi_n(z, \chi)$ are a maximum for those values of z for which the Brunt–Väisälä frequency is small. Since $\tilde{\Gamma}_n^0 = 0$ when there is no stratification, this term describes the effects of the potential flow of the homogeneous fluid around a source.[1]

The term $\tilde{\Gamma}_n^-$ is only non-zero when $\omega \neq 0$ and thereby determines the effects, localized close to the sources, of the transient nature of the IGW fields which are generated. The asymptotics of $\tilde{\Gamma}_n^-$ for large values of $|x|$, $|y|$ decline more slowly than $\tilde{\Gamma}_n^0$, which is due to the behaviour of the exponent in the integrand in the complex plane of the variable k and, in this case, the asymptotics of the term $\tilde{\Gamma}_n^-$ can be estimated by the method of steepest descents.

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